Development of the project: visualisations of transformations

Mette Andresen
e-mail: mette.andresen@math.uib.no
University of Bergen
Bergen, Norway

Abstract

This paper reports from a study of the ViT (Visualization of Transformations) project. The aim of the ViT project was to develop and test an interactive tool for linear algebra and publish it on the project's website http://kurs.uib.no/vit/ovelser.hml. *This study explored the teachers' implementation of the ViT project from a sociocultural perspective. The research question was: What driving forces and what obstacles can be identified for implementation of a concrete change towards integration of visualization tools in a linear algebra course? Data for the study includes interviews with the teachers and the rest of the project group, students' answers to two questionnaires and tasks from the linear algebra course.*

1. Introduction and background for this study

The department of Mathematics at University of Bergen gives a second year course in linear algebra for around 250 students from different programs in mathematics and science such as pure and applied mathematics, physics, geophysics, statistics etc. The course was previously taught almost completely in algebraic representation and in general, many students found it difficult and/or tedious, as well as difficult to cope with. The ViT project, running from 2009 to 2013 and funded by 'Norges Universitetet' aimed to improve the linear algebra course in collaboration with its lecturers and the groups of student assistant teachers (groups of 3 - 6 persons each year). The ViT project was meant to give students the opportunity to widen their view on linear algebra and to deepen their understanding of the involved mathematical concepts and relations. The project's means was to develop a visualization tool and to support the lecturers' and the students' use of the tool during the lectures. The visualization tool was created as a learning resource in the form of a web site with clusters of interactive sites (workshops), dealing with Vector spaces, Geometric Objects, Simple Transformations, and Determinants etc. (See http://kurs.uib.no/vit/).

The study reported in this article took place only during the last semester of the ViT project (spring 2012). It was based on data consisting of tasks and other teaching materials, evaluation sheets and interviews with the lecturers, the assistant teachers and with members of the ViT project group (5 interviews in all). Originally, our intention was to study how the potentials of visualisations were realised in the ViT project, by observing students using the active sites while working with their tasks. We hoped, thereby, to provide the sufficient basis for the interpretation of the role of the sites as a resource for learning. In that case, the study would demonstrate the potential of visualisation-tools in a linear algebra course in this particular context.

A preliminary evaluation of the ViT project was carried out in 2011 through a students' questionnaire which, however, had a lower response rate than expected. The evaluation showed that the students had little interest for the visualisations. This result corresponded with the lecturers' and the assistant teachers' impressions during all three rounds of the ViT. Considering the fact that very few students actually used the sites, we decided that it would be too difficult to carry out our study in accordance with the original plan. In contrast to the students' apparent lack of interest, the lecturers and the ViT project group found the idea of visualization of concepts and relations in linear algebra and the idea of interactive sites constructive and promising as a means to support and deepen the students' learning. This discrepancy, between the mathematicians' and didacticians' professional views on the one hand and the lack of interest and lack of efforts invested by the students on the other hand, was articulated in a new research question which became leading for the study presented here. Focus of our study was changed to these, apparently more pressing concerns, articulated in a revised research question: *What obstacles and what driving forces can be identified for implementation of a concrete change towards integration of visualization tools in a linear algebra course?*

2. Theoretical frame

The study in this article was based upon the interpretative framework for analysing individual and collective activity in classrooms developed by Paul Cobb et al. and described in [3] with modifications according to the actual organization of the course.
The theory of instrumental genesis was taken into account regarding technological aspects of the ViT project.

In order to establish a background for the discussion of the findings, results of recent research into potentials and successful conditions for teachers' professional development are outlined. Further, an ICMI study of the teaching and learning of linear algebra at university level, and a small qualitative study of introduction and use of ICT in school mathematics, are presented. Hence, the theoretical frame for the study in this article consists of five parts:

1. Cobb et al.'s interpretative frame for analysis including social and psychological perspectives, ref. [3],[10]
2. A summary of research based guidance for teachers' successful professional development, ref. [9], [1], [8]
3. The theory of instrumental genesis for the use of ICT, ref. [5], [6]
4. A small qualitative study of different approaches to the introduction of ICT in mathematics teaching, ref. [2]
5. An ICMI study of linear algebra at university level, ref. [4]

2.1 Social and psychological perspectives

The interpretative frame for analysis developed by P. Cobb et al. as presented in [3] coordinates both individual (psychological) and collective (sociological) perspectives. According to [10] the constructs of the framework, shown in Tabel 1, were developed from extensive classroom-based research with the aim to make sense of the students' learning in the classrooms. The term 'social norms' refers to regularities in the interaction patterns that regulate social interactions in the classroom, thus making the 'social norms' expressions of the normative expectancies in the classroom. For example, the interaction patterns in the classroom might be indicative of the expectation that students are capable of explaining their thinking to others. By contrast, the term 'socio-mathematical norms' refer to regularities in the interaction patterns that are specific to mathematics, for example norms regarding acceptable mathematical explanations, or norms regarding visualization as an argument. Furthermore, there is a reflexive relationship between the individual and the collective, meaning an analysis of norms must necessarily take the corresponding psychological components into account. Beliefs are to be taken as the psychological correlates of norms and are as such basically cognitive, meaning that discussions of norms and discussions of beliefs are intimately intertwined. The interrelationship between beliefs and norms is critical because it provides a means for talking theoretically about changes in beliefs. Beliefs can be thought of as an individual's understandings of normative expectancies. Social norms can be thought of as taken-as-shared beliefs that constitute a basis for communication and make a smooth flow of classroom interactions possible. ([10] pp 315-316).

Table 1

Social Perspectives	Psychological Perspectives
Classroom social norms	Beliefs about one's role, others' roles, and the general nature of mathematical activity in school
Socio-mathematical norms	Specifically mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions

For our study of the ViT project this framework offered a connection between the use of the visualization tools from the individual student's perspective on the one hand and on the other hand from the assistant teachers', the lecturers' and the ViT- project's didacticians' shared perspective. The connection was established in the following meaning: if beliefs are seen as the cognitive basis used by individuals to interpret situations arising in the course of social interaction, then the evolution of the students' beliefs over time may be studied by considering the reflexive relationship between beliefs and norms. The norms, further, are indicative of the students' beliefs. In this interpretation the implementation of the use of the visualization tools should imply development of social and socio-mathematical norms and mathematical practices, appropriate for its use. The implementation of the use of the tools should also imply development of the students' beliefs in accordance with visualizations, about what constitutes mathematical activity, as well as about their own role and about the teacher's role.

The teachers who were in charge of implementation of the use of the tools played an important role: In general, the teacher's own beliefs are essential for development and the implementation of norms and practices in the classroom with the students. In the ViT project the teachers were also, with the support of lecturer and the didacticians, in charge of the overall development of new norms which, in turn, were intended to be intertwined with the individual students' growing beliefs according to the framework represented in Table 1. From this perspective, the ViT project required a certain form of professional development from the lecturers and from the assistant teachers. Therefore, some relevant aspects of surveys of recent research about professional development were included in our study of the ViT project, in addition to the interpretative framework developed by Cobb et al. in [3], [10].

It should be noted that after the aforementioned revision of the research questions, our study did not encompass analysis of visualization as a means for the learning of linear algebra.

2.2 The ViT project in the perspective of professional development

In the article 'The mathematical education and development of teachers' in NCTM's Second Handbook, Judith T. Sowder addresses what it means to prepare teachers of mathematics and to support them with the professional learning opportunities they need in order to lead their students to succeed in learning mathematics [9]. Our study of the ViT project considered the assistant teachers on the one hand as professional teachers, but on the other hand as mathematics students, only temporarily employed with teaching. Hence, the paragraph in [9] regarding the question: '*What principles can be used to guide the design of professional development?*' was chosen as a suitable state-of-the-art background for the study of implementation of the use of ViT's visualisation tools in the classroom as a case of professional development for the assistant teachers, with the aim of shedding light on the conditions for implementation of visualisation tools in the linear algebra course. Sowder concluded that successful professional development reported in recent research had a great deal in common, especially:

1. the role of determining the purpose of a professional development program,
2. the role of teachers in deciding on foci,
3. the important role of collaborative problem solving
4. the necessity of modelling the type of instruction expected

In addition, professional development was more likely to produce enhanced knowledge and skills, if it (amongst others) was:

5. based on providing teachers opportunities for 'hands-on' work (active learning).

According to [1] these findings are also in accordance with the Scandinavian paradigm as it manifested itself in the report:' Hur kan lärare lära?' [8], prepared for NCM (The National Centre for Mathematics Education in Sweden). Consequently the key points 1. to 5. above were chosen as the basis of our analysis of the ViT project.

2.3 Use of technology

The introduction of technology use was a crucial element of the ViT project. Our study of the ViT project took as its starting point that in themselves, the interactive sites do nothing. According to the theory of *Instrumental genesis and instrumented techniques*, an artefact like for example the ViT sites do not in itself serve as a tool for anybody. It becomes useful, and then denoted an instrument, only after the user's formation of (one or more) mental utilisation scheme(s). Such utilisation schemes connect the artefact with the user's conceptual knowledge and with his or her understanding of the way it may be used to solve a given task. Thereby, the utilisation schemes contribute to the formation of instrumented action schemes. An instrument then consists of: i) the tool, for example the interactive sites, ii) the user's mental utilisation schemes and iii) the task or problem to be solved. [5] (p 96-97). The term instrumental genesis denotes the process in which the artefact becomes an instrument. [6] (pp 165-169). The formation of utilisation schemes and construction of instrumented action schemes proceed through activities in "The two-sided relationship between tool and learner as a process in which the tool in a manner of speaking shapes the thinking of the learner, but also is shaped by his thinking." [6] (p 190). In other words, in the ViT project the ViT sites would support for the students' learning only if the students managed to include them, mentally, into their strategies, routines and resources for problem solving. The challenge for the teachers, hence, was to create a learning environment where the ViT sites served to form a technical dimension in the students' mathematical activity, and to ensure that the activity would lead to useful visualisations of concepts and relations.

2.4 Two different approaches to the introduction of ICT tools

In [2] the authors discerned between two different approaches to the introduction of computer based tool into the classroom. The study built on observations of students' resistance against development of personal tools from ICT artefacts and their teachers' attempts to push this process. According to the theory of instrumental genesis the student must change his or her scheme of the mathematical problem situation including the mathematical concepts involved when developing a personal instrument. Hence, the students' resistance in the study might be explained, [2], by noticing that apart from being intellectually challenging, the process of creating personal instruments may also challenge one's perception of what mathematics is, i.e. the personal beliefs and norms. In order to implement ICT as part of the mathematical activities in the classroom the teachers in the study had to persistently insist on the students' use of it. Two different approaches to the introduction of a new tool, were identified in [2], and represented by teacher A and teacher B:

Teacher A allowed the new technology to be included in the class' problem work in a natural way. Not surprisingly, the appropriate use of this technology was rather sparse in the first lessons. Teacher A simulated an *adidactical* situation with regard to use of the tool, in the sense that focus was on strategies for solving tasks, on organising the group work and on the mathematical content, i.e. on calculus. Technology might be regarded simply as a part of the environment in teacher A's class. It was one option and solution strategy that the students might choose.

Teacher B, on the contrary, spent the first lessons with the new tool helping the students getting familiar with it. In teacher B's class the introduction was a *didactical* situation, meaning that the students were asked to work with and learn about the tool, and trust the teacher's claim that it would pay off in the long run.

Three factors would, in general, influence teachers' choice between A and B [2]:

The extent of the expected use of the tool. Teacher A expected the students to gradually progress in the process of instrumental genesis at his/her own pace in a self-regulating manner, ending up with incidents of tool use. In contrast, teacher B expected the students to fully integrate the tool in all their work. Teacher B thus, found that developing skills on their own would take too much of a bite out of the students' intellectual capacity in the actual setting.

The expected emergence of students' motivation. This raised the question whether it was the teacher's obligation to motivate the students for learning to use the tool, or if the existing social norms of the classroom would gradually develop and 'feed' the students' motivation.

The complexity of forced extra obligations related to different aspects of the introduction of ICT tools into the classroom. For example, technical obstacles and problems with hardware and software may force the teacher to change his or her teaching plans before or during the lessons. Teaching plans where the lessons are dependent of the tool seem to be much more fragile than plans where the tool is included 'on demand'.

These factors were highly relevant for our analysis (in 5.3 beyond) of the driving forces and obstacles related to technology use.

2.5 An ICMI study of Linear algebra at university level

In [4] DOrier and Sierpinska discuss recent of research into the teaching and learning of linear algebra. The discussion is organised along three axes: a) curriculum reform actions, b) analysing the sources of students' difficulties and c) research based and controlled teaching experiments.

Ad a) the authors [4] very briefly report that curriculum reforms, in USA based on the work of 'Linear Algebra Curriculum Study Group' LACSG but also in Europe, tends to be directed towards numerical computations, and towards taking into account real life implementations of linear algebra into account.

Ad b) The authors of [4] points to the common agreement that linear algebra is a cognitively and conceptually difficult subject. Some of the difficulty stems from the level of abstraction since the concept of a vector space is an abstraction from a domain of already abstract objects. From a didactical point of view, according to the authors, the problem is that any linear problem, within the reach of a first year university student, can be solved without using the axiomatic theory. Only the experts gain from abstractions in terms of unification, generalization and simplification. Further, they refer to research pointing to a single massive obstacle appearing for students, namely what they denote the obstacle of formalism within the theory of vector spaces, and the interpretation of the formal concepts, in relation with more intuitive contexts like geometry or systems of linear equations. They refer to a research study in which a teaching experiment was designed in linear algebra explicitly taking into account the question of *registers* (graphical, tabular and symbolic registers of the language of linear algebra), in particular the conversions from one register to another. The results not only showed that the students could be successfully trained to perform correct conversions but also that this training improved their linear algebra grades, though the improved grades might be a result of the extra training. Finally, the article points to another epistemological study of the connection between geometry and linear algebra. The aim of this study was to characterize the meaning from a didactical point of view of geometrical intuition in relation with linear algebra, according to various authors. The result of this study showed that the necessity of geometric intuition was: '*.. very often postulated by textbooks or teachers of linear algebra. However, in reality, the use of geometry was most often very superficial. (...) In other words, the geometrical reference acted as an obstacle to the understanding of general linear algebra. (...) It seems that the use of geometrical representations or language is very likely to be a positive factor, but it has to be controlled and used in a context where the connection is made explicit.'* [4]

Ad c) The studies reported in [4] were not conclusive. Concerning the teachers, the results of the experiment showed a great variability. Concerning the students, it gave an idea of what remained one, two or more years after the course.

From the short list of implications of research in [4], the following three are of special interest for our analysis (in 5.4 beyond) of driving forces and obstacles for the learning of linear algebra in the ViT project:

It must be recognized that a very large majority of students taking the undergraduate linear algebra courses are not preparing to specialize in mathematics, but in a variety of other domains and disciplines. The axiomatic approach in these courses is therefore highly questionable.

The success rate in linear algebra courses cannot be improved by obliging students to take courses in logic and set theory as prerequisites. In linear algebra, students have difficulties with the interpretation of the formal concepts of vector space theory in the more intuitive contexts of geometry and systems of linear equations, rather than with formalism in general.

Although geometric embodiments may help the students in understanding the more abstract concepts, it is not a good idea to start a linear algebra course with vector geometry and build the algebraic concepts as a generalization from geometry. The relation between linear algebra and geometry is, epistemologically, less natural than it may appear. In some cases, geometry may act as an obstacle to students' understanding of linear algebra.

3. Methods

Our study of the ViT project encompassed two components:

A background study into the goals and aims of the ViT project, required because the project had been running for two years at the time of our study. The background study inquired questions like: what norms and beliefs are targeted about visualisation and about the teaching and learning of linear algebra, and how and by whom did the project intend to develop them? The study also included the content of the website, for example testing whether it seemed to be of any help for non-experts of linear algebra.

An evaluation of the ViT part of the linear algebra course based on interviews with lecturers and assistant teachers and on students' questionnaires in 2011 and 2012

Data for analysis in the two parts was not completely separated. An interview with one of the lecturers for example, gave information about the aims and motives for the project but also about the insight from it.

Individual interviews with the 3 lecturers, group interview with 4 student teacher assistants and pair interview with two members from the ViT project group were carried out. The interviews were intended to provide i) concrete information about the ViT project, and insight into the participants' view on ii) the learning potentials of visualization in linear algebra, iii) the possibilities for changing university education towards this direction, and iv) the students' interests, abilities and motivation with regard to the ViT project.

Two questionnaires were distributed to the students in 2011 and 2012. The questions focused on the students' impression of the websites and their role in the course, and on their view on its usefulness. The first questionnaire (in 2011) was handed out at the course's last lecture, whereas the second questionnaire (in 2012) was handed out in connection with compulsory submission tasks. Even if

the respondent rates to both questionnaires were too low to obtain significant results, answers to the second questionnaire inspired and informed the interview with the student teacher-assistants.
The textbook (excerpts) [7], homework tasks, course plans and other teaching materials from the course served to illustrate the intended visualizations. Amongst these were specific tasks from the ViT project questioning visualizations and with reference to particular sites, created with the aim to encourage the students’ use of the sites. These tasks were included in the compulsory submissions from the students and as such, they served in our study to exemplify the lecturers’ attempts to guide the students.
Data was analysed in accordance with the theoretical framework comprised by the five blocks described in 2. (above).

4. Data

4.1 The Course materials

The following excerpt from the teaching materials, Exercise Set number 1 (spring 2012), illustrates the lecturer’s attempts to prompt the students to use the ViT website. The exercises intend to help the students to obtain a second view on elementary objects in linear algebra in addition to their traditional algebraic view. The Exercise Set links to one of the interactive ViT sites. The screen dumps of the sites (fig 1, fig 2, fig 3 and fig 4) were produced in our study to represent envisioned cases of students’ inquiries related to the exercise set (in the lack of authentic students’ answers).

Use the links: http://kurs.uib.no/vit/index.html for more details.

It should be noted that use of the web sites was optional during the course, never formally requested. The only exception was one set of homework task which included optional visualisation exercises. The students, apparently, were not at all forced to use the web sites.

Visualization Exercise Set 1:

Exercise 1

Go to workshop”2.2.3 four” at the Vit site
http://kurs.uib.no/vit/2explore/four/index.html. Choose a rectangle and study how it is mapped by different linear transformations. Look specifically at the following transformation matrices, and explain the mapping using words you know from geometry;

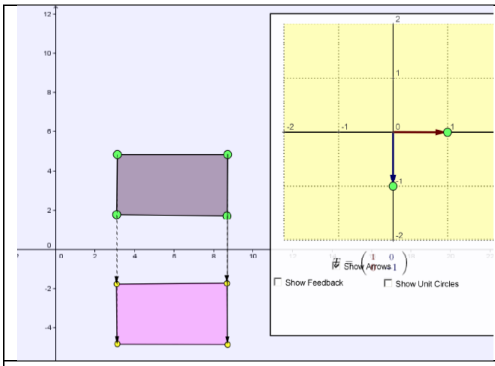
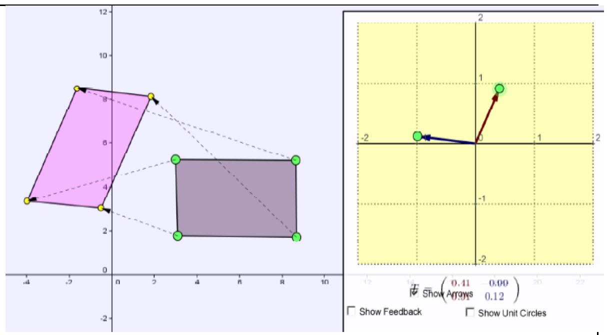
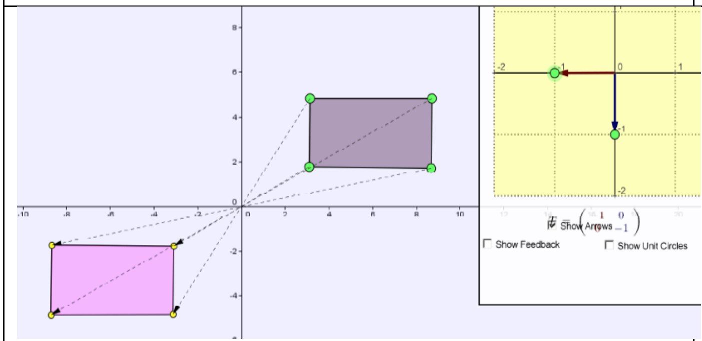
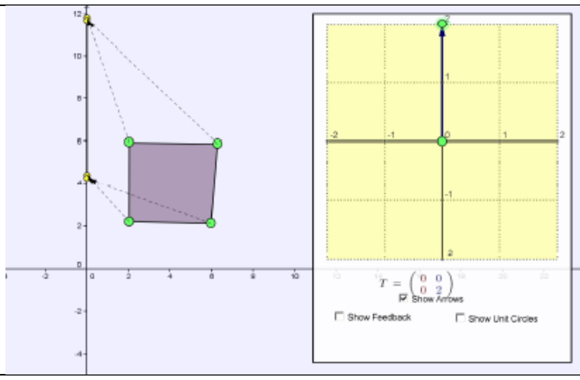
$$a)\,T_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad b)\,T_2=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\quad c)\,T_1=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

$$d)\,T_4=\begin{pmatrix}2&0\\0&2\end{pmatrix}\quad e)\,T_5=\begin{pmatrix}0.8&0\\0&0.8\end{pmatrix}\quad f)\,T_6=\begin{pmatrix}0&0\\0&2\end{pmatrix}$$

$$g)\,T_7=\begin{pmatrix}2&0\\0&3\end{pmatrix}$$

’.

Figure 1 (exercise 1)

	
Relates to a). The students may relate the matrix to a reflection in the x-axis	A random example which relates to study of the mapping of a rectangle by transformations.
	
Relates to c). The students may relate the matrix to reflection through origo	Relates to f). The students may relate the matrix to projection onto the y-axis

Exercise 2

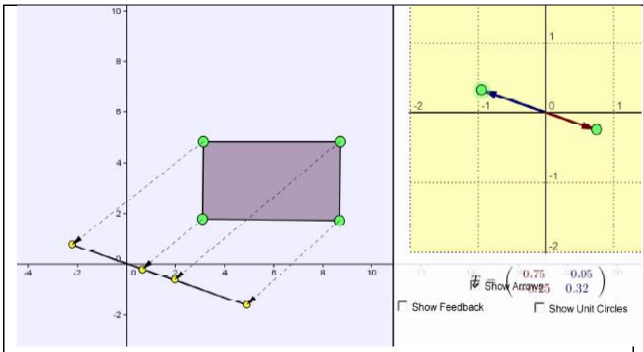
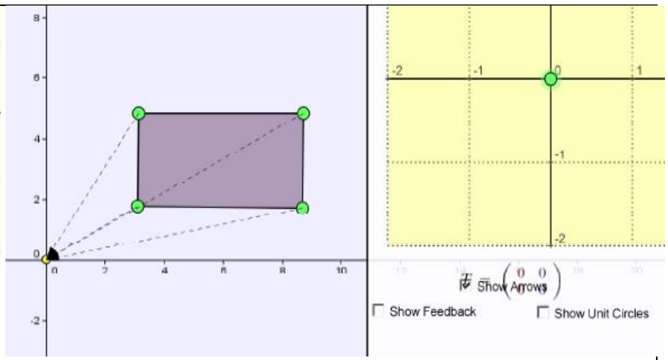
Choose a rectangle. Find transformation matrices that map the rectangle into;

a) a triangle, (Is it possible?)

b) a line,

c) one point. (All corners at the same point).

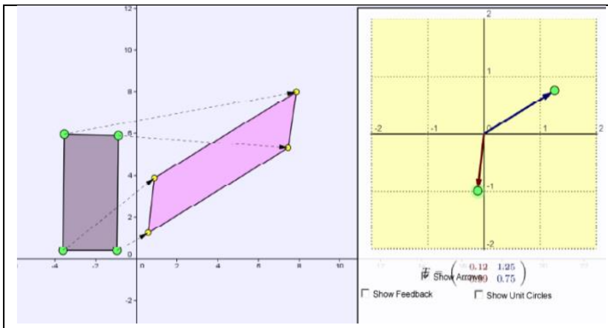
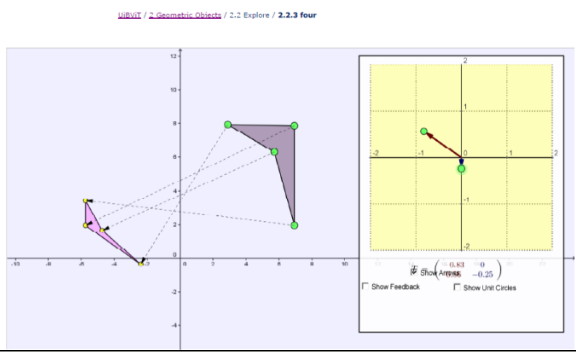
Figure 2 (exercise 2)

	
Relates to b)	Relates to c)

‘Exercise 3

Move one of the corners of a rectangle along its diagonal until it passes the other diagonal. The result is a concave rectangle. Follow the mapping of the resulting rectangle during a transformation, and try to explain the mapping of concave and convex rectangles.’

Figure 3 (exercise 3)

	
Random example of mapping of a convex rectangle	Random example of mapping of a concave rectangle

‘Exercise 4

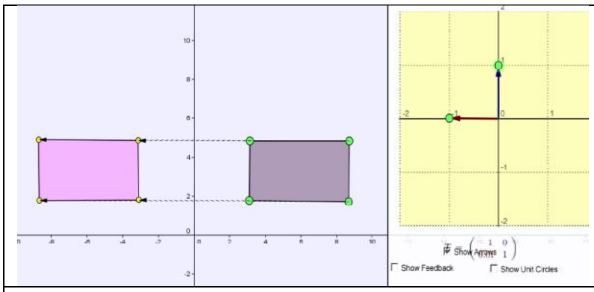
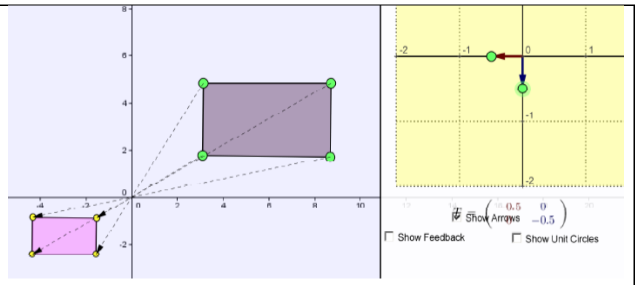
How will you choose the entries in the transformation matrix if the effect of the mapping is to be a

a) reflection through the y-axis?

b) reflection through the origin?

c) reflection through origin with a simultaneous contraction to one-third of the size?’

Figure 4 (exercise 4)

	
Relates to a) This is similar to what might be result of b) in the first exercise. But this time the basis vectors of the matrix are subject of change during the students' experiments	Relates to c), basis vectors are subject of change during the students' experiments

4.2 The textbook

The textbook [7] follows the recommendations of LACSG ('Linear Algebra Curriculum Study Group', USA). Visualisation as a means to support the students' concept formation was integrated into the textbook. In the preface linear transformations are mentioned under Distinctive Features (of the textbook):

> 'Early introduction of key concepts. Many fundamental ideas of linear algebra are introduced within the first seven lectures, in the concrete setting of Rn, and then gradually examined from different points of view. Later generalisations of these concepts appear as natural extensions of familiar ideas, visualised through the geometric intuition developed in Chapter 1. A major achievement of this text is that the level of difficulty is fairly even throughout the course' (...)
>
> 'Linear transformations form a 'thread' that is woven into the fabric of the text. Their use enhances the geometric flavour of the text. In Chapter 1, for instance, linear transformations provide a dynamic and graphical view of matrix-vector multiplication' [7]
>
> Under Pedagogical Features is further mentioned:
>
> 'A. Strong Geometric Emphasis. Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualise an idea. There are substantially more drawings here than usual, and some of the figures have never before appeared in a linear algebra text.' [7] p preface xi)

In general, the textbook is offering a broad selection of applications that illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science and others. The textbook is supplied with support for technology use in the forms of a study guide with introductory material for work with MATLAB, Maple, Mathematica a.o., and files with data for about 900 numerical exercises in the text, Case studies and application projects. Fig. 5 shows two examples of visualisation in excerpts from the text.

Figure 5: examples of visualisation in the course's textbook)

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of **u** and **v** under multiplication by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, **v**. ■

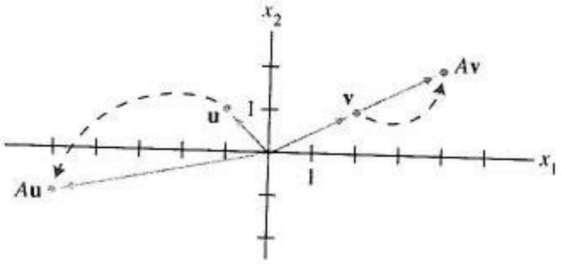
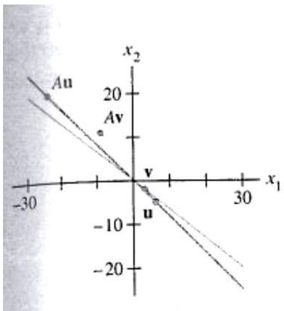


FIGURE 1 Effects of multiplication by A.

Lay (2012) p. 266



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are **u** and **v** eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus **u** is an eigenvector corresponding to an eigenvalue (−4), but **v** is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of **v**. ■

[10] p 267

The textbook's view on visualisations which was stressed in the introduction and represented by example 1 and 2 is included in our analysis of driving forces and obstacles related to the ViT project's norms and beliefs in 5.1.

4.3 Interviews

Students' motivations were discussed in an interview with one of the creators of the interactive sites (C) on 13th January 2012. According to C the major part of the students did not feel motivated to spend time on the activities on the website because they couldn't see how the activities were related to the tasks they had to solve at the final examination. On the other hand, the visual competence may be regarded as an indispensable part of the students' mathematical competence within this field. Further, C had the impression that doing the website's activities might be hard and troublesome for the students because of 'the thinking'. If the examples in the textbook caused troubles some of the students were tempted to skip them and concentrate only on the text. In a similar situation, they might be tempted to avoid use of the ViT sites, even when a site was easy to handle from a technical point of view.

In an interview 2nd February 2012 with the lecturer (L) from the course in linear algebra in 2009 and 2010, L explained his view on the use of visualisation tools in linear algebra. L pointed the problem out that most students' ideas and conceptions in linear algebra are closely related to numbers and calculations whereas they have little feeling with, for example, the concept of an eigenvector. L found the sites particularly useful when introducing a new concept or idea, for illustration and as a means to understand 'what it really was about'. According to L, the content of the sites and the visualisations would be of relevance for about 2/3 of the course.

In general, the students were obliged to give in, and pass, five out of seven sets of written tasks. Only one of these seven sets requested use of the interactive sites – meaning that the students could choose not to use the sites at all and still pass fulfil the formal requests. L referred to a concrete example of a task set where the students had to choose between a very difficult, theoretical task and a very easy one based on use of the sites. Most of the students choose the difficult one.

In the interview 1st June 2012 with 4 student assistant-teachers and 1 lecturer three main issues were discussed:
a) The ViT sites' finish
b) The students' choices; to use or not to use the sites
c) Potentials in general of visualisations in linear algebra

Ad a) The ViT sites did not appear finished. The assistant teachers gave critical comments and referred to concrete examples of sites that did not work and sites with errors, wrong texts, undeleted comments etc. None of the assistant teachers though had the impression that the students' lack of interest was caused by the unfinished look. When the content of the sites was discussed the group found that better introduction to the purpose and use of the individual site was needed. The assistant teachers found that even if a student might be able to find the meaning of a site and its connection with the course on his own hand, the sites would still be of little help because it lacked the appropriate explanations. One of the assistant teachers suggested, supported by the others, that the interactive sites in a more finished form with detailed introductions might stimulate the students' interest if the sites were presented at the lectures. The students might then start to use the sites. One of the assistant teachers, though, expressed his view on mathematics as a handcraft subject where technical tools could easily disturb the students and distract them from the core ideas.

b) The lecturer and the assistant teachers told about the common agreement that recommendations for the students at the course were explicitly given at two levels: need to do and nice to do. The students had the opportunity to choose between two levels of ambitions when completing the course (and in their study): They could choose to do as little as possible by completing the routine tasks and train for the written examination, or they could aim at understanding, do extra tasks and try to widen their understanding of concepts and relations for example supported by the ViT sites. This common agreement was shared by the other lecturers, according to the individual interviews. One of the assistant teachers argued, and the others agreed, that 90% of the students taking the course had no interest in understanding the content or interest in mathematics at all, since they were studying biology, economy etc. These students would find the visualization tool of interest only if they either i) had the experience that it was really a shortcut to pass the exam or ii) if they were 'forced' to use it, i.e. if the tasks for the examination requested the use of it. Use of the ViT web site during the written examination, on the other hand, would cause troubles since the students were not allowed even to use computers then, not to mention having internet access. One assistant teacher proposed that the visualisation could be introduced during the lectures and incorporated into the tasks, but assessed independently of the ViT website. The students might be requested to make drawings for illustration and explanation, using the ViT website during the semester but only by the use of paper and pencil at the examination.

c) What are the potentials of visualization in linear algebra – if any? There was agreement in the group that within certain topics, for example *eigenvalues*, visualization should offer good help for understanding and application. Apparently, the meaning of visualisation was understood to be mapping between representations and change between representations like in the exercises 1 – 4 (Fig. 1, 2, 3 and 4). Nobody disagreed in the overall statement that visualization was desirable in one form or another. Nobody went into details, neither, about the meaning of the term visualisation, neither did anybody explain exactly what benefits or advantages visualisation was supposed to give the students.

5. Analysis of data: Driving forces and obstacles

5.1 Driving forces and obstacles related to the ViT project's norms and beliefs

In the following, the aims and goals of the ViT project are described in terms of norms and beliefs using the social and psychological perspectives in Cobb et al.'s interpretative framework [3], [10] and Table1. Each perspective of the ViT project's norms and beliefs is then discussed with reference to the relevant data.

Seen from social perspectives, a successful implementation of the ViT tool in the linear algebra course would imply development in the classrooms of

1s. Mathematical practices in the form of mapping objects, principles and relations in graphical representation with the corresponding ones in formal algebraic representation.

2s. Socio-mathematical norms concerning the changes between representations and for the use of different media.

3s. Social norms regarding the individual students' investment of time and efforts in his or her study

Ad1s. These new mathematical practices were in accordance with the practice outlined by the lecturers in the interviews and by the textbook, for example: Example 1 (Figure 5 in paragraph 4.2 above) contains an illustration of multiplications of vectors u and v, talking about to stretch or dilate the vectors; in Example 2, the sketch or illustration of the problem serves as support the argument for the solution. The agreement with regard to the mapping of representations between practices of the textbook and the lecturers on the one hand and the ViT sites on the other, must be seen as a driving force for implementation of the use of the ViT sites even though development of these mathematical practices did not set implementation of the ViT sites as a prerequisite.

Ad2s. The socio-mathematical norms for changes between representations were not purely dictated by the ViT project, since the textbook and the lecturers already included these changes independent of the project. But the use of computer was not integrated into the linear algebra course. Computer use was not an issue outside the ViT part of the course, and the interviews clearly revealed that the major part of the students never brought their computers to the lectures or the classes, apparently because they did not use them in linear algebra or maybe in mathematics at all, neither at university nor at home. Getting used to use the interactive sites would be a demanding and time consuming process which should include several cases of instrumental genesis. In this case the instrumental genesis would imply much time and efforts invested by the students. The instrumental genesis therefore counts as a major barrier or obstacle to a successful implementation of the ViT tool.

Ad3s. The atmosphere on the course was good as well as the relations amongst the students and between the students and their teachers and lecturers. Neither the interviews nor the evaluation sheets gave an impression, for example, of ambitious competitive students under hard pressure. Apparently, any student would be respected for his or her interest, engagement or labour. In itself the need of introducing social norms prescribing a vast investment of time, labour and efforts, consequently, should be considered neither a driving force nor an obstacle to a successful implementation of the ViT tool.

From psychological perspectives, the students were supposed to end up, accordingly, with

1p. Flexible mathematical conceptions in linear algebra with regard to graphical and algebraic representation

2p. Representational competence within transformations, meaning the ability to choose appropriate representations of transformations depending on the context (task or problem, explanation or understanding). The ability to choose appropriate medium, i.e. computer, paper & pencil, blackboard & chalk, talk etc. should be included as one part of the representational competence

3p. Since use of the ViT interactive sites was introduced as an option for those who wanted to acquire a deeper understanding of linear algebra, implementation would only take place by students who saw themselves as 'upper class' students with interest in, maybe even talent for linear algebra.

Ad1p. Although the flexibility (or lack of it) was made more visible in the ViT part of the course this does not mean that flexibility was imposed by the ViT project as an extra learning goal. The flexibility of the students' mathematical conceptions as such was not evaluated but as long as the linear algebra course in general, not only the ViT project, intended to encourage and support the students' development of flexible mathematical conceptions, the desire for flexibility must be interpreted as a strong driving force for implementation of use of the ViT sites.

Ad2p. The desire for representational competence might be interpreted as a driving force based on similar arguments as for the flexible conceptions. Concerning the ability to choose appropriate medium, though, the implementation of computer use without help with the instrumental genesis would be a far too heavy task for the students to manage. As long as the students were supposed to manage the instrumental genesis as an 'invisible' job included in their study of visualisations in linear algebra, the demand of (computer) representational competence must be interpreted as a major obstacle.

Ad3p. Like for 3s (above), the students' personal view on themselves as 'upper class' students in linear algebra should not in itself be interpreted neither as an obstacle nor a driving force. According to the interviews the social norms would allow each student to individually choose his or her status and position with regard to ambitions and interest depending on skills, capability and attitudes.

5.2 Driving forces and obstacles for ViT as professional development

During the ViT project, the lecturers giving the linear algebra course were in favour of implementing the ViT tools. This was also in agreement with the textbook's approach to linear algebra; they collaborated with the ViT project group from the early phase of the project, and influenced the content and design of the sites. Thus for the lecturers the overall conditions for successful implementation were very good, following [9]. During the interview, the assistant teachers took a positive attitude towards the idea of visualisation in linear algebra. The group of assistant teachers, though, were not involved with the design of the sites, neither with decisions concerning the content of the sites or content of the course in general. They did not reveal any shared engagement in the implementation of the interactive sites.

During the interview the assistant teachers pointed to the facts that:

i) The students seldom brought their laptop to the classroom neither on the linear algebra course nor any other courses.

ii) Many students did not see linear algebra, or mathematics, as their main interest of study. They only took the course because they were obliged to do so when educating in other subject like biology etc.

iii) There was a lack of alignment between the idea of implementing computer based visualisation into the course on the one hand, and the requested hand-writing at the written examination

iv) Use of the ViT sites was not necessary, not to say helpful, for solving the major part of the homework tasks

These obstacles (discussed in the previous paragraph on social and psychological perspectives and in the following paragraph on instrumental genesis) were acknowledged by the lecturers and also by the ViT project group. There was a common agreement between assistant teachers and lecturers that any student with interest and engagement could still benefit from the investment of time and efforts if he or she wanted to use the ViT sites. The lecturer and the assistant teachers would not perceive it as a problem, neither as a failure, if the major part of the students chose not to use the ViT sites.

According to the main results of [9] implementation of the ViT sites should then, expectedly, only succeed as an optional activity for the students.

5.3 Driving forces and obstacles related to technology use

The instrumental genesis which, according to [5],[6], was needed for the sites to become useful for the students had no place neither in the course nor in the ViT project as such. Apparently, the students were more or less expected to get familiar with the ViT sites on their own hand. According to the interviews the sites intended to appear self-explanatory even if not all the interviewees agreed in this assumption.

The Visualization Exercise Set 1 illustrates that this would not necessarily be the case. The screenshots show the independent (brown) and the dependent (pink) quadrilateral, whereas the transformation is represented by basis vectors in the circle. It is completely up to the user to try this out and to interpret the results on his, or her, own hand. The exercise set's tasks might serve to initiate the process of instrumental genesis by asking leading questions and encourage the students' investigations. In our interpretation, the progression in the series of exercises 1 – 3 follows two directions. The exercises are getting more and more open ended, besides getting more and more complex. The double progress can be seen where the students, after the 'warming up' section in exercise 1, are supposed to consider a number of particular transformations of one single rectangle whereas in exercise 3, all transformations of any four-sided shape can do. The students, thereby, are encouraged to make inquiries on their own hand of the correlations.

Exercise 4 (Fig. 4) served to make the students map the different representations in the inverse direction of the mapping in exercise 1. The visualisation supported by the interactive sites related to exercise 1 – 4 was in the form of changes in both directions between graphical and formal, algebraic representations, and by mapping between mathematical objects and relations which appeared in these two representations.

Both the double progress and the two way mappings between representations can serve well as support of the important processes of instrumental genesis. The content of the exercises also serve to link the visual impressions to the abstract concepts which, apparently, should be in the kernel of visualisation. Hence, these active sites could in the best case become first step towards an instrument for visualisation when the student had finished his or her work with the optional set.
The introduction of the use of ICT is closely linked to the potentials of the process of instrumental genesis. In the ViT case the teachers and the lecturer, apparently, resemble teacher A in [2]: The students were not expected to fully integrate the sites into their work with the linear algebra course even if nothing prevented them from using the sites to a large extent if desired. The lecturer and the teachers expected that motivation for using the ViT sites would emerge by some of the students, gradually over time when they experienced their usefulness. This expected motivation should lead the students to learn, on their own hand, to use the sites.

5.4 Driving forces and obstacles for the learning of linear algebra

Students' well known difficulties with interpretation of the formal concepts of vector space theory in the more intuitive contexts of geometry and systems of linear equations were addressed in the ViT project. According to recent research surveyed in [4] visualisation in the form of linking linear algebra to geometrical representation is very likely to be positive, if it is well controlled and used in a context where the connection is made explicit. This was clearly the case when/if using the ViT sites. Following [4] it is important to take into account that many students on linear algebra courses do not specialise in mathematics - which was done in the ViT project by giving students the options to operate at 'low level'.

6. Conclusion and perspectives

In general, the ViT project fulfilled important criteria for success summarised in [9] and [4]. There was a high degree of alignment of content of the textbook, exercises, lectures and classroom teaching: the textbook claimed to give strong emphasis on geometric interpretation of every major concept. The lecturers had stressed the importance of visualisation too, with and without computer, but visualisation completely without computer was an option of no interest for this study of the ViT project. This was in line with the ViT project's ideas and aims, and the mathematical content was coherent. The collaboration between the project group, the lecturers and the assistant teachers was good and there was a good atmosphere during the lectures and the classroom teaching.

6.1 Computer use was the main obstacle

Our study of the ViT project showed that the main barrier or obstacle was the computer use. The obstacles appeared at different levels and in different perspectives:

1. Lack of alignment between the idea of implementing computer based visualisations into the course on the one hand, and the requested hand-writing at the written examination
2. Socio-mathematical norms for changes between different media including computer use.
3. Specifically mathematical beliefs and values particularly concerning the use of computer and the choice between media

Ad 1.: Lack of alignment: Request of documentation of use of computer related to some problem or task concerning visualisation could easily be implemented in the form of one or two obligate homework tasks.

Ad 2 + 3: The role of visualisation in the ViT project was internal mathematical, as a means for basic, geometrical interpretation of the linear transformations; visualisation was not for example introduced as a means for arguing, or a tool for problem solving. Neither was visualisation seen as shortcut to solution of other tasks or exercises during the course. In this sense, visualisation was seen as mathematical activity suitable to support the students' formation of certain mathematical conceptions. Since linear algebra is commonly acknowledged to be a cognitively and conceptually difficult area [2] the visualisation was thereby linked with the touchiest spot, so to say, of the course. Therefore it seems reasonable to have let the students have the opportunity to avoid deeper engagement if they wanted to, for example those who studied other subjects than mathematics. On the other hand, it was clear from our study that the potentials are correspondingly huge which might suggest that all students should be obliged to go through a tutorial and prove at a certain level that they had generated one or two sites as an *instrument* in accordance with the theory of instrumental genesis.

Whether the use of ViT sites should be optional or not, the assistant teachers would be in charge of developing the corresponding competencies. So, advices or support would be needed to help the assistant teachers to include into their teaching the generation of the sites as their own and, successively, the students', instruments: Though being a necessary prerequisite, the assistant teachers' knowledge about the sites' features and facilities as well as the potentials of these for visualisation in the linear algebra course, moreover, would still not be sufficient to guide the students. The assistant teachers would need insight into and personal experience with the generation of a tool from the ViT sites, for example in the form of a teachers' guide combined with one or two seminars with hands on activities and training. Tasks like the exercises 1 – 4 would be of good help with regard to students' process of instrumental genesis.

7. References

[1] Andresen, M. and Henriksen, B. Effective, long-range developments – evaluation of a national initiative. Report on the Mathematics Project, 2006 – 2008. NAVIMAT, Copenhagen. ISBN: 978-87-91786-00-6, 2010
[2] Andresen, M. and Misfeldt, M. Teachers' dealing with technology-enhanced mathematics in teaching practice. In: Yang, W.-C, Majewski, M., deAlwis, T., and Cao, Y. (eds.) *A journey to discover more mathematics – Proceedings of the fourteenth Asian Technology Conference in Mathematics, 17.-21. Dec 2009.* Beijing Normal University, Beijing, China. ISBN 978-0-9821164-2-5 (print) 1940-4204 (online). Pp 286-295, 2009
[3] Cobb, P. Conducting Teaching Experiments in collaboration with teachers. In Kelly, A.E. and Lesh, R. (eds.). *Handbook of Research Design in Mathematics and Science Education*. Pp 307-333.Mahwah, NJ, Lawrence Erlbaum, 1999
[4] Dorier, J.-L. and Sierpinska, A. Research into the teaching and learning of linear algebra. In: Holton, D. (ed.). *The teaching and learning of mathematics at university level: An ICMI study*, 255-273. Kluwer Academic Press Netherlands, 2001
[5] Drijvers, P.:"Learning Algebra in a computer environment. Design research on the understanding of the concept of parameter" dissertation, Utrecht, CD-beta press, 2003
[6] Drijvers, P. & Gravemeijer, K. Computer Algebra as an Instrument. In Guin, D., Ruthven, K. & Trouche, L. *The Didactical Challenge of Symbolic Calculators* Springer. Pp 163-196, 2005
[7] Lay, David C. *Linear Algebra and its applications*. Fourth edition. Addison – Wesley, 2012
[8] Mouwitz, L. Hur kan lärare lära? – Internationella erfarenheter med focus på matematikutbildning (How can teachers learn? – International experiences focused on mathematics education) Nationellt Centrum för Matematikutbildning, Göteborg, Sweden, 2001
[9] Sowder, J.T. The Mathematical Education and Development of Teachers. In Lester, F.K. (ed.). *Second Handbook of Research on Mathematics Teaching and Learning*. Pp. 157 - 223. National Council of Teachers of Mathematics. Information age Publishing. USA, 2007
[10] Yackel, E. and Rasmussen, C. Beliefs and norms in the mathematics classroom. In: Leder, G.C., Pehkonen, E. and Törner, G. (eds.). *Beliefs: A hidden variable in mathematics education?* Pp 313 – 330. Kluwer Academic Press, 2002